ICERM Lecture 3

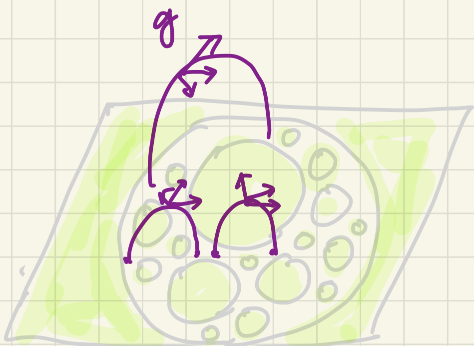
(geodesic planes in
∞-vol hyp mflds)

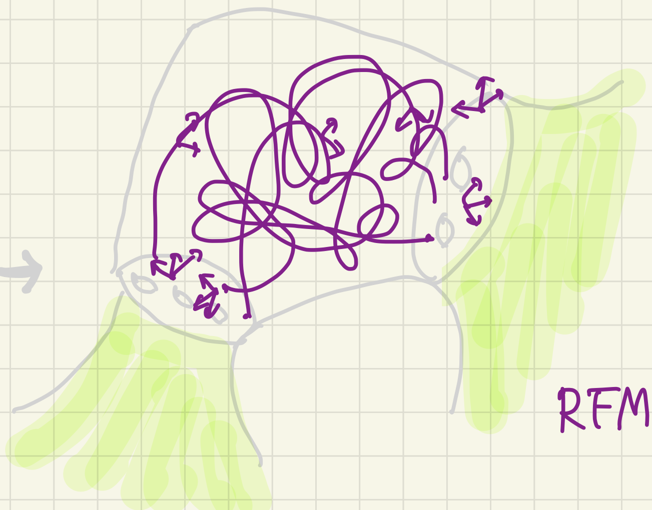
Hee Oh
(Yale University) May 17, 2023

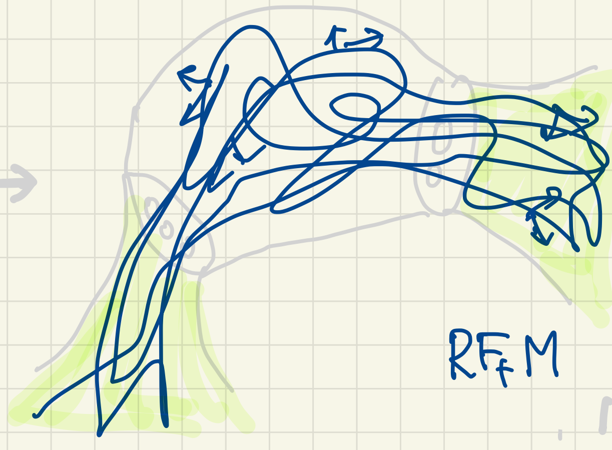
$G = SO^\circ(n,1) = \mathrm{Isom}^+(\mathbb{H}^n)$ $\quad n \geq 3$

$\Gamma < G$ convex cocpt

$RFM = \{[g] \in \Gamma\backslash G \mid g^{\pm} \in \Lambda\} \subset RF_+M = \{[g] \mid g^+ \in \Lambda\} \subset \Gamma\backslash G$







$$N = \left\{ \begin{pmatrix} 1 & x & * \\ & I & x^t \\ & & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{R}^{n-1} \right\}$$ max unipotent subgp $< G$

$U \cong \mathbb{R}^{k-1} < N$ conn.

$H(U) = \langle U, U^t \rangle \simeq SO^\circ(k,1)$

$\overline{xH(U)}$? $\overline{xU}$?

any reductive subgp $\supset H(U)$

$$\mathcal{L}_{H(U)} = \{ L = H(\hat{U})C \mid U < \hat{U} < N$$
$$z\, H(\hat{U})C \text{ closed for some } z \in RF_+M \}$$

any reductive subgp $\supset U$

$$\mathcal{L}_U = \{ vLv^{-1} \mid L \in \mathcal{L}_U,\ v \in N \}$$

Thm (MMO $n=3$, LO $n\geq 4$)

$M = \Gamma\backslash\mathbb{H}^n$ CC hyp mfld with Fuchsian ends

(1) $H(U)$-orbit closures

$\forall x \in RFM$,

$$\overline{xH(U)} = xL \cap RF_+M \cdot H(U)$$

for some $L \in \mathcal{L}_{H(U)}$

(2) U-orbit closures

$\forall x \in RF_+M$,

$$\overline{xU} = xL \cap RF_+M$$

for some $L \in \mathcal{L}_U$

(3) Equidistributions

$x_i L_i$ max. closed orbits $\quad x_i \in RF_+M$, $L_i \in \mathcal{L}_U$

$$\lim_{i\to\infty} x_i L_i \cap RF_+M = RF_+M$$

For $i = 1, 2, 3,$

$(i)_m$ holds if (i) is true for all $U < N$ with codim $\leq m$

$m=0$ (1) & (3) trivial

(2) minimality of N-action

$$
\begin{aligned}
(2)_m + (3)_m &\Rightarrow (1)_{m+1} + (2)_m + (3)_m \\
&\Rightarrow (1)_{m+1} + (2)_{m+1} + (3)_m \\
&\Rightarrow (2)_{m+1} + (3)_{m+1}
\end{aligned}
$$

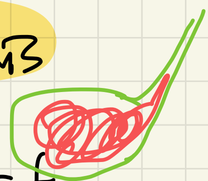
<u>Rmk</u> If $\mathrm{vol}(M) < \infty$, $(3)_m$ is not needed in the induction pf.

What is special about CC mflds with Fuchsian ends?

"Recurrence of $U = \{u_t \mid t \in \mathbb{R}\}$-orbits"

If $\Gamma\backslash G$ cpt, xU remains in a cpt subset

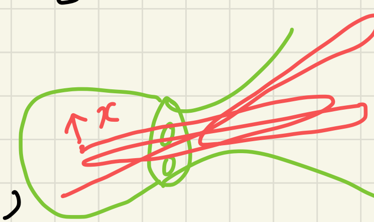
If $\mathrm{vol}(\Gamma\backslash G) < \infty$, Dani-Margulis



$\forall x \in \Gamma\backslash G$, & $\forall \varepsilon > 0$, $\exists$ cpt $\mathcal{C}$ s.t.

$$\frac{1}{2T}\left|\{t \in [-T, T] \mid \underline{xu_t} \in \mathcal{C}\}\right| \geq (1-\varepsilon)$$

If $\mathrm{vol}(\Gamma\backslash G) = \infty$, for a.e x,



$\forall$ cpt $\mathcal{C} \subset \Gamma\backslash G$,

$$\frac{1}{2T}\left|\{t \in [-T, T] \mid xu_t \in \mathcal{C}\}\right| \to 0$$

Prop $M = \Gamma \backslash \mathbb{H}^n$ Fuchsian ends

$\exists\, k > 1$ s.t $\forall x \in RFM$,

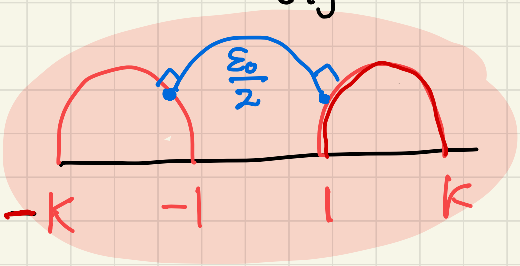
$T(x) = \{ t \in \mathbb{R} \mid x U_t \in RFM \}$ is k-thick,

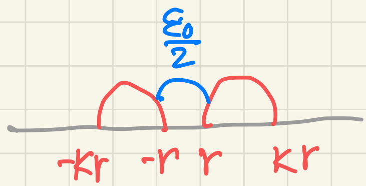
i.e, $\forall\, r > 0$,

$T(x) \cap \left([-kr, -r] \cup [r, kr] \right) \neq \phi$

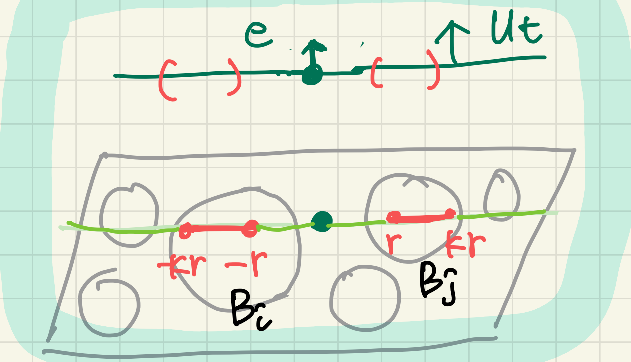
$S^2 - \Lambda = \bigcup B_i$

Pf) $\varepsilon_0 = \inf_{i \neq j} d(\mathrm{hull}\, B_i, \mathrm{hull}\, B_j)$





$0, \infty \in \Lambda$ $\qquad U = \mathbb{R} = \{x_1\text{-axis}\}$



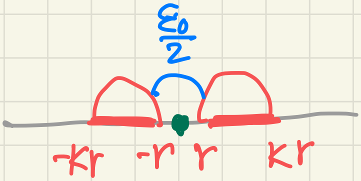
$\{ t \mid [e] U_t \in RFM \}$

$\parallel$

$\{ t \mid U_t^- = t \in \Lambda \}$

$= \mathbb{R} \cap \Lambda$

6

$d_{\mathbb{H}^2}\Big($ 

$\Big) \ngeq d_{\mathbb{H}^3}(\text{hull } B_i, \text{hull } B_j)$

$\rotatebox{90}{$\geq$}$

$*$ $\qquad \varepsilon_0$

Unipotent blowup lemma

Let $g_n \to e$ in $G - N(U)$

$T \subset \mathbb{R} \cong U$ $\quad$ K-thick subset

$\limsup\limits_{n\to\infty} T g_n U$ contains $q \in N(U) - U$.

$d(q, e) < \varepsilon$

Spse $y u_n \to x \in \overline{yU}$

$y u_n = x g_n$

$g_n \notin N(U)$

$g_n \to e$

Since $T(x) = \{ t \in \mathbb{R} = U \mid x u_t \in RFM \}$

is k-thick,

$\exists\ u_{t_n} \in T(x)$ and $u_{s_n} \in U$

s.t $\quad u_{t_n} g_n u_{s_n} \longrightarrow q \in N(U) - U.$

$$x g_n u_{s_n} = \underbrace{x u_{t_n}}_{\in RFM} (u_{-t_n} g_n u_{s_n})$$

$$\rightarrow z q \quad \in \quad \overline{yU}$$

This is good enough for $n=3$

to show $\quad P$ is closed or dense

in $\Gamma \backslash \mathbb{H}^3$

In higher dim ≥ 4, we also need

Avoidance principle

$\mathcal{G}(U) = \{x \in RF_+M \mid xU$ is

not contained in any

closed orbit of $L \subsetneq G\}$

$\mathcal{S}(U) = RF_+M - \mathcal{G}(U)$

Want: If $x \in \mathcal{G}(U)$, $\overline{xU} = RF_+M$

For this, we need to understand

$T^*(x) = \{t \in \mathbb{R} \mid xu_t \in RFM$

$xu_t \notin \mathcal{S}(U)\}$

Avoidance Thm (Lee-O.)

$M = \Gamma \backslash \mathbb{H}^n$ Fuchsian ends.

$\exists$ cpt subsets $E_1 \subset E_2 \subset \cdots$ s.t

$$\mathscr{G}(U) \cap RFM = \bigcup_{i \geq 1} E_i \quad \text{s.t}$$

$\forall x \in \mathscr{G}(U) \cap RFM$, $\forall i \geq 1$,

$\exists$ open $\mathcal{O}_i \supset E_i$ s.t

$$T^{**}(x) = \{ t \in \mathbb{R} \mid x u_t \in RFM - \mathcal{O}_i \}$$

is k_0-thick.

Thank You !!